

Characterization of Mean Residual Life for Uniform and Exponential Distributions

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Abstract

In this material, starting from relationships between failure rate, mean residual life and cumulative density function, we obtain a simple expression for mean residual life for each classical continuous distribution function. In this case we analyse cases of uniform and exponential distributions.

Key words: *distribution, failure rate, mean residual life*

A delicate problem in the research of reliability theory is the precarious amount of information which the analyst has for using. Taking into consideration that this pursuit can be done, and from a certain moment, then a special importance is given to mean residual life. The aim of this paper consists in finding the expression for mean residual life starting from the classic repartitions of reliableness and also in identifying the primary calculus for the data and information we have available. Starting from [1], the authors are trying to determine the simplified relations of the calculus. The results obtained for each demonstration are based on the knowledge of differential equations and in the respective elements of analytics mathematics.

One can employ characterization of mean residual life for uniform and exponential distributions user variables without having to store them in:

X_x – a random variable truncated on the left $X_x = (X - x \mid X \geq x)$

$e(x)$ – mean residual life of X_x truncated on the left $e(x) = M(X - x \mid X \geq x)$

$m(x)$ – mean function of X_x $m(x) = (X \mid X \geq x)$

$r(x)$ – failure rate of X_x

c_1, k – constants.

Using relationship between $m(x)$ and $e(x)$: $m(x) = (X \mid X \geq x) = e(x) + x$.

Is known: $m(x) = k + q(x) \cdot r(x)$ and $r(x) = \frac{m'(x)}{m(x) - x} = \frac{e'(x) + 1}{e(x)}$.

Then $m(x) = k + q(x) \cdot \frac{m'(x)}{m(x) - x} \Leftrightarrow m^2(x) - x \cdot m(x) = k \cdot m(x) - k \cdot x + q(x) \cdot m'(x)$.

or

$$\begin{aligned} m'(x) \cdot q(x) + m(x) \cdot (k + x) &= m^2(x) + k \cdot x \\ \Leftrightarrow m'(x) + m(x) \frac{k + x}{q(x)} &= \frac{m^2(x)}{q(x)} + \frac{k \cdot x}{q(x)}. \end{aligned}$$

For a cumulative density function for Weibull, normal, student, Laplace, power, uniform and exponential distributions when $k=0$, obtain a Bernoulli equation. In other words:

$$m'(x) + m(x) \frac{x}{q(x)} = \frac{m^2(x)}{q(x)}.$$

By a straightforward change of variables $m(x) = z^{-1}(x)$ or $m'(x) = z^{-2}(x) \cdot z'(x)$ we have a linear differential equation:

$$-\frac{z'(x)}{z^2(x)} + \frac{x}{q(x)} \cdot \frac{1}{z(x)} = \frac{1}{q(x)} \cdot \frac{1}{z^2(x)}$$

or

$$-z'(x) + \frac{x}{q(x)} \cdot z(x) = \frac{1}{q(x)}.$$

In other words, we have: $z'(x) - \frac{x}{q(x)} \cdot z(x) = -\frac{1}{q(x)}$.

Then we find this solution for equation:

$$z(x) = c_1 \cdot e^{\int_{x_0}^x \frac{u}{q(u)} du} + \int_{x_0}^x \frac{-1}{q(s)} \cdot e^{\int_{x_0}^s \frac{t}{q(t)} dt} ds.$$

If $z(0) = \frac{1}{m(0)} \stackrel{Not}{=} m_0 = M(X | X \geq 0)$ in some cases is mean of random variable.

Then we have the following expansion for the mean residual life using $\int \frac{u}{q(u)} du$.

(c) Uniform distribution:

$$q(x) = \frac{1}{2}(b^2 - x^2), b \in \mathfrak{R}. x \in (a, b)$$

$$\int \frac{u}{\frac{1}{2}(b^2 - u^2)} du = -2 \int \frac{u}{b^2 - u^2} du = \int \frac{-2u}{b^2 - u^2} du = -\int \frac{2u}{u^2 - b^2} du = -\ln|u^2 - b^2|.$$

Then, in this case, we have:

$$\begin{aligned}
 z(x) &= c_1 \cdot e^{\int_{x_0}^x \frac{u}{q(u)} du} + \int_{x_0}^x \frac{-1}{q(s)} \cdot e^{\int_{x_0}^s \frac{t}{q(t)} dt} ds = c_1 \cdot e^{-\ln|u^2-b^2| \Big|_{x_0}^x} - \int_{x_0}^x \frac{2}{b^2-s^2} \cdot e^{-\ln|t^2-b^2| \Big|_s^x} ds \\
 &= c_1 \cdot \left| \frac{x_0^2-b^2}{x^2-b^2} \right| - \int_{x_0}^x \frac{2}{b^2-s^2} \cdot \left| \frac{x_0^2-b^2}{x^2-b^2} \right| ds = c_1 \cdot \left| \frac{x_0^2-b^2}{x^2-b^2} \right| - \int_{x_0}^x \frac{2}{b^2-s^2} \cdot \frac{s^2-b^2}{x^2-b^2} ds = \\
 &= c_1 \cdot \frac{b^2-x_0^2}{b^2-x^2} - \frac{2(x-x_0)}{b^2-x^2} = \frac{c_1(b^2-x_0^2)-2(x-x_0)}{b^2-x^2}.
 \end{aligned}$$

When $x_0 = a$ the result obtained is:

$$z(x) = \frac{c_1(b^2 - a^2) - 2(x - a)}{b^2 - x^2} \quad \text{and} \quad z(a) = \frac{c_1(b^2 - a^2) - 2(a - a)}{b^2 - a^2} = \frac{c_1(b^2 - a^2)}{b^2 - a^2} = c_1.$$

But $m(a) = \frac{a+b}{2}$ and $z(a) = \frac{2}{a+b}$.

In other words:

$$z(x) = \frac{\frac{2}{a+b} \cdot (b^2 - a^2) - 2(x - a)}{b^2 - x^2} = \frac{2(b-a) - 2(x-a)}{b^2 - x^2} = \frac{2(b-x)}{b^2 - x^2} = \frac{2}{b+x}.$$

We have: $m(x) = \frac{1}{z(x)} = \frac{b+x}{2}$ and $e(x) = m(x) - x = \frac{b+x}{2} - x = \frac{b-x}{2}$.

Then a definite result for this distribution is: $e(x) = \frac{b-x}{2}$.

(d) Exponential distribution:

$$f(x) = ae^{-ax}, \quad a > 0, \quad x \geq 0.$$

Because this relation is valid

$$\frac{f'(x)}{f(x)} = \frac{k-x-q'(x)}{q(x)},$$

its determination of $q(x)$ is reduced to the solution of differential equation:

$$q'(x) - aq(x) = k - x$$

which has the solution:

$$q(x) = c \cdot e^{ax} + \int_{x_0}^x (k-s)e^{as} ds \quad \text{with } c \in \mathfrak{R}.$$

When $x_0 = 0$ the result obtained is:

$$q(x) = e^{ax} \cdot \left(C - \frac{1}{a^2} + \frac{k}{a} \right) + \left(\frac{1}{a^2} + \frac{x}{a} - \frac{k}{a} \right).$$

Or, one of its reduced forms: $q(x) = \frac{1}{a}x + \frac{1}{a^2}$.

For the moment, this can be determined:

$$z(x) = c_1 \cdot e^{\int_0^x \frac{a^2 u}{au+1} du} + \int_0^x \frac{-a^2}{as+1} e^{\int_0^s \frac{a^2 t}{at+1} dt} ds \text{ with } c \in \mathfrak{R}.$$

But

$$m(0) = \frac{1}{a}.$$

Then, a result for exponential distribution is:

$$z(x) = \frac{a}{ax+1} \text{ or } m(x) = \frac{ax+1}{a}.$$

This means: $e(x) = \frac{1}{a}$.

References

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Caracterizarea duratei de viață reziduală pentru distribuțiile normale și exponențiale

Rezumat

În cadrul acestui articol, pornindu-se de la relațiile de legătură dintre densitatea de probabilitate $f(x)$, rata de defectare $r(x)$ și media reziduală a duratei de funcționare $e(x)$, sunt determinate relații din care se pot deduce relații analitice de calcul pentru $e(x)$. Ipotezele pornesc de la 1., unde sunt determinate relații simplificate de calcul pentru găsirea densităților de probabilitate $f(x)$. Toate rezultatele găsite sunt însoțite de demonstrații bazate pe cunoștințe de ecuații diferențiale și respectiv elemente de analiză matematică.